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Nonresonant smoothing in Sobolev spaces for coupled wave + transport equations

Christophe Pallard

D.M.A., École normale supérieure, 45, rue d'Ulm, 75230 Paris cedex 05, France

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Abstract

Consider a system consisting of a linear wave equation coupled to a transport equation:

$$\Box_{t,x} u = f,$$

$$(\partial_t + v(\xi) \cdot \nabla_x) f = P(t, x, \xi, D_\xi) g,$$

where $P(t, x, \xi, D_\xi)$ is a linear differential operator of order m in ξ. Such a system is called *nonresonant* when the maximum speed in the transport equation is less than the propagation speed in the wave equation. Velocity averages of solutions to such nonresonant coupled systems are shown to be more regular than those of either the wave or the transport equation alone. This question was investigated first in terms of Sobolev spaces H^s in the paper of F. Bouchut, F. Golse and C. Pallard, *Non-resonant smoothing for coupled wave + transport equations and the Vlasov–Maxwell system*, (Rev. Mat. Iberoamericana, 2003, in press.) The same authors also studied a related question in *On classical solutions to the 3D relativistic Vlasov–Maxwell system: Glassey–Strauss' theorem revisited* (Arch. Rational Mech. Anal., in press). Here we state a result in Sobolev spaces $W^{s,p}$. More precisely, if f, g belong to $L^p_{\mathrm{loc}}(\mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R}^M)$ and with initial data for u regular enough, then for any test function $\chi \in \mathcal{C}^m_c(\mathbb{R}^M_\xi)$ we show that

$$\int u(\cdot, \cdot, \xi) \chi(\xi) \, d\xi \in W^{1+\gamma, p}_{\mathrm{loc}}\bigl(\mathbb{R}^*_+ \times \mathbb{R}^N\bigr),$$

when $\gamma = 1 - (N-1)|\frac{1}{2} - \frac{1}{p}| \geqslant 0$ and $1 < p < +\infty$. We also study the limit cases $p = 1$ and $p = +\infty$ when $N = 3$.

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E-mail address: pallard@dma.ens.fr (C. Pallard).

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1. Introduction

We shall consider the following system defined on $\mathbb{R}_+^* \times \mathbb{R}_x^N \times \mathbb{R}_\xi^M$,

$$\begin{aligned} &\Box_{t,x} u = f, \\ &\big(\partial_t + v(\xi)\cdot\nabla_x\big) f = P(t,x,\xi,D_\xi) g, \end{aligned} \tag{1.1}$$

where $\Box_{t,x} = \partial_t^2 - \Delta_x$ is the wave operator, $P(t,x,\xi,D_\xi)$ a linear differential operator in the ξ variable, $v \equiv v(\xi)$ a smooth vector field $\mathbb{R}_\xi^M \to \mathbb{R}_x^N$ and $g \equiv g(t,x,\xi)$ a given function. The Cauchy problem for (1.1) is to find a couple of real-valued functions $u \equiv u(t,x,\xi)$ and $f \equiv f(t,x,\xi)$ defined for $(t,x,\xi) \in \mathbb{R}_+^* \times \mathbb{R}_x^N \times \mathbb{R}_\xi^M$ and satisfying the initial conditions

$$u_{|t=0} = u_I, \qquad \partial_t u_{|t=0} = u_I', \qquad f_{|t=0} = f_I. \tag{1.2}$$

The aim of this paper is to study in terms of Sobolev spaces the local regularity of averages

$$\rho_\chi(t,x) = \int u(t,x,\xi)\chi(\xi)\,d\xi,$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}_\xi^M)$ is a test function. We shall make the further assumption that v satisfies the *nonresonant* condition:

$$\sup_{\xi\in\operatorname{supp}\chi} \big|v(\xi)\big| < 1. \tag{NR}$$

Then our main result is the following.

Theorem 1.1. *Let $1 < p < +\infty$ such that $(N-1)|\frac{1}{2} - \frac{1}{p}| \leqslant 1$. Suppose that f and g belong to $L^1_{\mathrm{loc}}(\mathbb{R}_\xi^M, L^p_{\mathrm{loc}}(\mathbb{R}_+^* \times \mathbb{R}_x^N))$. Assume that the initial data satisfy $f_I \in L^p_{\mathrm{loc}}(\mathbb{R}_+^* \times \mathbb{R}_x^N)$, $u_I' \in L^1_{\mathrm{loc}}(\mathbb{R}_\xi^M, W^{1,p}_{\mathrm{loc}}(\mathbb{R}_x^N))$ and $u_I \in L^1_{\mathrm{loc}}(\mathbb{R}_\xi^M, W^{2,p}_{\mathrm{loc}}(\mathbb{R}_x^N))$. Let $P(t,x,\xi,D_\xi)$ be a linear differential operator of order $m \in \mathbb{N}$ on $\mathbb{R}_\xi^M$ with smooth coefficients. Pick a test function $\chi \in C_c^m(\mathbb{R}_\xi^M)$ and let $v \in C^m(\mathbb{R}_\xi^M)$ satisfy the nonresonant condition* (NR). *Then, if* (1.1), (1.2) *hold, the ξ-average*

$$\rho_\chi(t,x) = \int u(t,x,\xi)\chi(\xi)\,d\xi$$

belongs to $W^{1+\gamma,p}_{\mathrm{loc}}(\mathbb{R}_+^* \times \mathbb{R}_x^N)$ where $\gamma = 1 - (N-1)|\frac{1}{2} - \frac{1}{p}|$.

In dimension $N = 3$, the above theorem still holds in the limit cases $p = 1$ and $p = +\infty$.

Theorem 1.2. *Let $p = 1$ or $p = +\infty$. Suppose that f and g belong to $L^1_{\mathrm{loc}}(\mathbb{R}_\xi^M, L^p_{\mathrm{loc}}(\mathbb{R}_+^* \times \mathbb{R}_x^3))$. Assume that the initial data satisfy $f_I \in L^p_{\mathrm{loc}}(\mathbb{R}_+^* \times \mathbb{R}_x^3)$, with $u_I' \in L^1_{\mathrm{loc}}(\mathbb{R}_\xi^M, W^{1,p}_{\mathrm{loc}}(\mathbb{R}_x^3))$ and $u_I \in L^1_{\mathrm{loc}}(\mathbb{R}_\xi^M, W^{2,p}_{\mathrm{loc}}(\mathbb{R}_x^3))$. Let $P(t,x,\xi,D_\xi)$ be a linear differential operator of order $m \in \mathbb{N}$ on $\mathbb{R}_\xi^M$ with smooth coefficients. Pick a test function $\chi \in C_c^m(\mathbb{R}_\xi^M)$ and* *let* $v \in C^m(\mathbb{R}^M_\xi)$ *satisfy the nonresonant condition* (NR). *Then, if* (1.1), (1.2) *hold, the* ξ*-average*

$$\rho_\chi(t,x) = \int u(t,x,\xi)\chi(\xi)\,d\xi$$

belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^*_+ \times \mathbb{R}^3_x)$.

Such a coupling between a wave equation and a transport equation arises naturally when studying the relativistic Vlasov–Maxwell system. In this context v is the relativistic velocity

$$v(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}},$$

which of course satisfies the nonresonant condition (NR). For more background and references, we refer to [1].

The plan of this paper is as follows. In Section 2 we prove Theorem 1.1. We proceed essentially as in [1], where the case $p = 2$ is addressed. The main difference is that we cannot use the energy estimate for the wave equation. Section 3 is devoted to the proof of Theorem 1.2, which requires a specific treatment based on properties of the elementary solution of the wave equation. In Section 4 we give two examples showing that Theorem 1.1 is sharp.

2. Proof of Theorem 1.1

2.1. L^p multipliers

The following result about L^p multipliers will be useful. It may be found in Stein's monograph [4].

Theorem 2.1. *Let* $m \in \mathcal{C}^l(\mathbb{R}^n \setminus \{0\})$ *be a bounded function with* $l > n/2$. *Assume that*

$$\left|\partial_k^\alpha m(k)\right| \leqslant A|k|^{-|\alpha|}, \quad \text{for all } |\alpha| \leqslant l. \tag{2.1}$$

Then the operator $T : L^2 \cap L^p \to L^2$ *defined for* $1 < p < +\infty$ *by* $\widehat{T\phi}(k) = m(k)\widehat{\phi}(k)$ *satisfies*

$$\forall \phi \in L^2 \cap L^p \quad \|T\phi\|_{L^p} \leqslant C\|\phi\|_{L^p},$$

where the constant C *depends only on* A, p *and* n.

We shall encounter another kind of multipliers, related to the wave equation. The next result was established in [3].

Theorem 2.2. *Let* A *and* B *be defined in* $\mathbb{R}^N$ *by*

$$\widehat{A\phi_0}(k) = \cos\bigl(|k|\bigr)\widehat{\phi_0}(k), \qquad \widehat{B\phi_1}(k) = \frac{\sin(|k|)}{|k|}\widehat{\phi_1}(k).$$

If we assume $1 < p < +\infty$ *and* $(N-1)|\frac{1}{2} - \frac{1}{p}| \leqslant 1$, *then we have*

$$\|A\phi_0\|_{W^{\gamma,p}} \leqslant c_0\|\phi_0\|_{W^{1,p}}, \qquad \|B\phi_1\|_{W^{\gamma,p}} \leqslant c_1\|\phi_1\|_{L^p}, \tag{2.2}$$

with $\gamma = 1 - (N-1)|\frac{1}{2} - \frac{1}{p}|$.

2.2. An elliptic operator

The key argument in the proof of Theorem 1.1 is that some well chosen combinations of the wave operator $\Box_{t,x}$ and of the transport operators $T^{\pm} = \partial_t \pm v(\xi) \cdot \nabla_x$ are elliptic in the variables t and x.

Lemma 2.1. *For* $\chi \in C_c^m(\mathbb{R}_\xi^M)$, *let* $v \equiv v(\xi)$ *in* $C^m(\mathbb{R}_\xi^M)$ *satisfy the nonresonant condition* (NR), *and let* $\lambda \in \mathbb{R}$. *The two following conditions are equivalent*:

- λ *satisfies the condition*

$$v_M^2 < \lambda < 1, \quad \textit{where } v_M = \sup_{\xi \in \operatorname{supp}\chi} \left|v(\xi)\right|; \tag{2.3}$$

- *for each* $\xi \in \operatorname{supp}\chi$, *the second order differential operator*

$$Q_\xi^\lambda = \lambda\Box_{t,x} - \left(\partial_t - v(\xi)\cdot\nabla_x\right)\left(\partial_t + v(\xi)\cdot\nabla_x\right) \tag{2.4}$$

  is elliptic.

When λ *verifies any of these conditions, the symbol* q_ξ^λ *of the operator* Q_ξ^λ *satisfies the following uniform estimates*: *for all multi-index* $\alpha \in \mathbb{N}^{1+N}$ *and* $\beta \in \mathbb{N}^M$ *such that* $|\beta| \leqslant m$,

$$\sup_{\xi\in\operatorname{supp}\chi}\ \sup_{\omega^2+|k|^2>0} \left(\omega^2+|k|^2\right)^{(2+|\alpha|)/2} \left|\partial_{\omega,k}^\alpha \partial_\xi^\beta \left(\frac{1}{q_\xi^\lambda(\omega,k)}\right)\right| < +\infty. \tag{2.5}$$

Proof. The symbol $q_\xi^\lambda(\omega,k) = \lambda(-\omega^2 + |k|^2) + (\omega - v\cdot k)(\omega + v\cdot k)$ is a homogeneous function of order 2 of the Fourier variables (ω,k). Notice that a ξ derivative does not affect this property, so that:

$$\partial_\xi^\beta\left(\frac{1}{q_\xi^\lambda(\omega,k)}\right) = \frac{N_\xi^\lambda(\omega,k)}{q_\xi^\lambda(\omega,k)^{2^m}}$$

with $N_\xi^\lambda(\omega,k)$ homogeneous of order $2^{m+1}-2$. Then

$$\partial_{\omega,k}^\alpha \partial_\xi^\beta\left(\frac{1}{q_\xi^\lambda(\omega,k)}\right) = \frac{P_\xi^\lambda(\omega,k)}{q_\xi^\lambda(\omega,k)^{2^{m+|\alpha|}}},$$

with $P_\xi^\lambda(\omega,k)$ homogeneous of order $2^{m+|\alpha|+1} - 2 - |\alpha|$. Hence

$$\begin{aligned}
&\sup_{\omega^2+|k|^2>0} \left(\omega^2+|k|^2\right)^{(2+|\alpha|)/2} \left|\partial_{\omega,k}^\alpha \partial_\xi^\beta \left(\frac{1}{q_\xi^\lambda(\omega,k)}\right)\right| \\
&\quad = \sup_{\omega^2+|k|^2=1} \left|\partial_{\omega,k}^\alpha \partial_\xi^\beta \left(\frac{1}{q_\xi^\lambda(\omega,k)}\right)\right|.
\end{aligned} \tag{2.6}$$

Besides, the Cauchy–Schwarz inequality implies the following lower bound for $q_\xi^\lambda(\omega,k)$:

$$(1-\lambda)\omega^2+\lambda|k|^2-(v\cdot k)^2 \geqslant (1-\lambda)\omega^2+\lambda|k|^2-|v|^2|k|^2.$$

If (2.3) holds, then

$$m_\lambda=\min\Big(1-\lambda,\inf_{\xi\in\operatorname{supp}\chi}\big(\lambda-\big|v(\xi)\big|^2\big)\Big)>0.$$

Therefore $q_\xi^\lambda(\omega,k)\geqslant m_\lambda(\omega^2+|k|^2)$, and (2.6) gives:

$$\sup_{\omega^2+|k|^2>0}\big(\omega^2+|k|^2\big)\left|\partial_{\omega,k}^\alpha\partial_\xi^\beta\left(\frac{1}{q_\xi^\lambda(\omega,k)}\right)\right|\leqslant\frac{1}{m_\lambda^{2^{m+|\alpha|}}}\sup_{\omega^2+|k|^2=1}\big|P_\xi^\lambda(\omega,k)\big|.$$

Since the right-hand side of the inequality above depends continuously on ξ, we infer the result (2.5) for any compactly supported function χ. Conversely, when (2.3) is not satisfied, it is obvious that the operator Q_ξ^λ is not elliptic for some $\xi\in\operatorname{supp}\chi$. $\square$

2.3. L^p regularity for the wave equation

Lemma 2.2. *Consider the Cauchy problem*:

$$\left[\begin{array}{ll}\square_{t,x}u=f, & (t,x,\xi)\in\mathbb{R}_+^*\times\mathbb{R}^N\times\mathbb{R}^M,\\ u_{|t=0}=u_0, & (x,\xi)\in\mathbb{R}^N\times\mathbb{R}^M,\\ \partial_t u_{|t=0}=u_1, & (x,\xi)\in\mathbb{R}^N\times\mathbb{R}^M,\end{array}\right. \tag{2.7}$$

where

$$f\in L^1_{\mathrm{loc}}\big(\mathbb{R}_\xi^M,L^p_{\mathrm{loc}}\big(\mathbb{R}_+^*\times\mathbb{R}_x^N\big)\big)$$

and with initial data

$$u_0\in L^1_{\mathrm{loc}}\big(\mathbb{R}_\xi^M,W^{1,p}_{\mathrm{loc}}\big(\mathbb{R}_x^N\big)\big)\quad\textit{and}\quad u_1\in L^1_{\mathrm{loc}}\big(\mathbb{R}_\xi^M,L^p_{\mathrm{loc}}\big(\mathbb{R}_x^N\big)\big).$$

Then when $|1/2-1/p|\leqslant 1/(N-1)$, *we have*

$$u\in L^1_{\mathrm{loc}}\big(\mathbb{R}_\xi^M,W^{\gamma,p}_{\mathrm{loc}}\big(\mathbb{R}_+^*\times\mathbb{R}_x^N\big)\big),$$

where $\gamma=1-(N-1)|\frac{1}{2}-\frac{1}{p}|$.

Proof. Consider first the usual initial value problem for the wave equation

$$\left[\begin{array}{ll}\square_{t,x}u=\psi, & (t,x)\in\mathbb{R}_+^*\times\mathbb{R}^N,\\ u_{|t=0}=\phi_0, & x\in\mathbb{R}^N,\\ \partial_t u_{|t=0}=\phi_1, & x\in\mathbb{R}^N,\end{array}\right. \tag{2.8}$$

where data (ψ,ϕ_0,ϕ_1) are test functions. We define the operators A_t and B_t:

$$\widehat{A_t\phi_0}(k)=\cos\big(t|k|\big)\widehat{\phi_0}(k),$$

$$\widehat{B_t\phi_1}(k)=\frac{\sin(t|k|)}{|k|}\widehat{\phi_1}(k).$$

The solution to (2.8) is given by

$$u(t,x) = A_t\phi_0(x) + B_t\phi_1(x) + \int_0^t B_s\big[\psi(t-s,\cdot)\big](x)\,ds.$$

Theorem 2.2 says that when $(N-1)|\frac{1}{2}-\frac{1}{p}| \leqslant 1$,

$$\|A_1\phi_0\|_{W^{\gamma,p}} \leqslant c_0\|\phi_0\|_{W^{1,p}}, \qquad \|B_1\phi_1\|_{W^{\gamma,p}} \leqslant c_1\|\phi_1\|_{L^p}, \tag{2.9}$$

with $\gamma = 1-(N-1)|\frac{1}{2}-\frac{1}{p}|$. Now observe that

$$\widehat{A_t\phi}(k) = \cos\big(t|k|\big)\widehat{\phi}\left(t\frac{k}{t}\right) = t^N\cos\big(t|k|\big)\widehat{\phi_t}(tk) = t^N\widehat{A_1\phi_t}(tk),$$

where we use the notation $\phi_\lambda(\cdot) \equiv \phi(\lambda\cdot)$. Thus we get $A_t\phi = (A_1\phi_t)_{1/t}$ and similarly $B_t\phi = t\cdot(B_1\phi_t)_{1/t}$. It follows from (2.9)

$$\|A_t\phi_0\|_{W^{\gamma,p}} \leqslant C_0(t)\|\phi_0\|_{W^{1,p}}, \qquad \|B_t\phi_1\|_{W^{\gamma,p}} \leqslant C_1(t)\|\phi_1\|_{L^p},$$

with C_0 and C_1 in $L^\infty([0,T])$. We have proved that if $\phi_0 \in W^{1,p}$ and $\phi_1 \in L^p$ then $A_t\phi_0$ and $B_t\phi_1$ belong to $L_t^\infty W_x^{\gamma,p}([0,T]\times\mathbb{R}^N)$ for any $T>0$. Assuming $\psi \in L^p(\mathbb{R}_+^*\times\mathbb{R}^N)$ we infer that the same result holds for the inhomogeneous part of u and for any $T>0$ we get

$$\sup_{t\in[0,T]}\big\|u(t,\cdot)\big\|_{W_x^{\gamma,p}} \leqslant C(T)\big(\|\psi\|_{L^p_{t,x}} + \|\phi_0\|_{W_x^{1,p}} + \|\phi_1\|_{L_x^p}\big). \tag{2.10}$$

Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}_t^+)$ be a test function. We want to show that $\theta u \in W_{t,x}^{\gamma,p}(\mathbb{R}_+^*\times\mathbb{R}^N)$. We write

$$\Box_{t,x}(\theta u) = \theta'' u + 2\theta'\partial_t u + \theta\Box_{t,x}u = 2\partial_t(\theta' u) - \theta'' u + \theta\psi.$$

Observe that $\Box_{t,x} = \Delta_{t,x} - 2\Delta_x$. It comes then

$$(I-\Delta_{t,x})(\theta u) = (I-2\Delta_x)(\theta u) - 2\partial_t(\theta' u) + \theta'' u - \theta\psi.$$

Take the Fourier transform of the previous equality. We get[1] in the sense of tempered distributions:

$$\widehat{\theta u} = \frac{1+2|k|^2}{1+\omega^2+|k|^2}\widehat{\theta u} + \frac{2\mathrm{i}\omega}{1+\omega^2+|k|^2}\widehat{\theta' u} + \frac{1}{1+\omega^2+|k|^2}\big(\widehat{\theta'' u} - \widehat{\theta\psi}\big).$$

We consider $(1+\omega^2+|k|^2)^{\gamma/2}\widehat{\theta u}(\omega,k)$. The estimate (2.10) ensures that θu and $\theta' u$ belong to $L_t^p W_x^{\gamma,p}$. For the two first terms it is then enough to see that Theorem 2.1 implies that the following functions are L^p multipliers:

$$\frac{1+2|k|^2}{(1+\omega^2+|k|^2)^{1-\gamma/2}(1+|k|^2)^{\gamma/2}},$$

and

$$\frac{\mathrm{i}\omega}{(1+\omega^2+|k|^2)^{1-\gamma/2}(1+|k|^2)^{\gamma/2}}.$$

[1] (ω,k) are the Fourier variables corresponding to (t,x).

For the two last terms we have also that $\theta'' u \in L^p_t W^{\gamma,p}_x$ and we know that $\theta\psi \in L^p_{t,x}$ so that eventually $(I - \Delta_{t,x})^{\gamma/2}(\theta u) \in L^p_{t,x}$ and

$$\|\theta u\|_{W^{\gamma,p}_{t,x}} \leqslant C\big(\|\phi_0\|_{W^{1,p}_x} + \|\phi_1\|_{L^p_x} + \|\theta\psi\|_{L^p_{t,x}}\big).$$

We now return to the Cauchy problem (2.7). Pick a test function $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^M_\xi)$. When $\xi \in \mathbb{R}^M$ is fixed, we can apply the previous estimate and integrate:

$$\|\chi\theta u\|_{L^1_\xi W^{\gamma,p}_{t,x}} \leqslant C\big(\|\chi u_0\|_{L^1_\xi W^{1,p}_x} + \|\chi u_1\|_{L^1_\xi L^p_x} + \|\chi\theta f\|_{L^1_\xi L^p_{t,x}}\big).$$

It remains to see that one needs only local regularity for the data. This is an immediate consequence of the finite speed propagation property of the wave operator. $\square$

2.4.

For an arbitrary λ, we have:

$$Q^\lambda_\xi u = \lambda\Box_{t,x} u - T^-_\xi T^+_\xi u.$$

The wave equation in (1.1) gives $\lambda\Box_{t,x} u = \lambda f$. Now if we merge the two relations in the system (1.1), we get:

Lemma 2.3. *Suppose that* (u, f, g) *satisfy* (1.1) *with null initial conditions on* u*. We note* $P(t, x, \xi, D_\xi)\phi = \sum_{|\alpha|\leqslant m} \partial^\alpha_\xi (a_\alpha(t, x, \xi)\phi)$*, and define* h_α *as the solution of the Cauchy problem*

$$\left[\begin{array}{ll} \Box_{t,x} h_\alpha = a_\alpha g, & (t, x, \xi) \in \mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R}^M, \\ h_{|t=0} = 0, & (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^M, \\ \partial_t h_{|t=0} = 0, & (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^M. \end{array}\right.$$

Define also h^I *as the solution of*

$$\left[\begin{array}{ll} \Box_{t,x} h^I = 0, & (t, x, \xi) \in \mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R}^M, \\ h^I_{|t=0} = 0, & (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^M, \\ \partial_t h^I_{|t=0} = f_I, & (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^M. \end{array}\right.$$

Then we have $T^+_\xi u = \sum_{|\alpha|\leqslant m} \partial^\alpha_\xi h_\alpha + h^I$.

Proof. This is a consequence of the uniqueness of the solution to the Cauchy problem for the wave equation. See [1] for details. $\square$

2.5. Proof of Theorem 1.1

We assume that N and p satisfy the assumptions of Theorem 1.1. Start with considering the case of vanishing g and f_I. Obviously the transport equation plays no role in this situation, and in view of the degree of regularity imposed on the initial data u_I and u'_I, we deduce from Lemma 2.2 that

$$u \in L^1_{\mathrm{loc}}\big(\mathbb{R}^M_\xi, W^{1+\gamma,p}_{\mathrm{loc}}\big(\mathbb{R}^*_+ \times \mathbb{R}^N\big)\big).$$

Hence we get the desired result:

$$\rho_\chi \in W^{1+\gamma,p}_{\mathrm{loc}}\big(\mathbb{R}_+^* \times \mathbb{R}^N\big).$$

In the sequel we shall assume that $u_I \equiv 0$ and $u_I' \equiv 0$. It follows from Lemma 2.3 that

$$Q_\xi^\lambda u = \lambda f - \sum_{|\alpha| \leqslant m} T_\xi^- \partial_\xi^\alpha h_\alpha - T_\xi^- h^I.$$

We write:

$$Q_\xi^\lambda u = \lambda f - \sum_{|\alpha| \leqslant m} \big[T_\xi^-, \partial_\xi^\alpha\big] h_\alpha - \sum_{|\alpha| \leqslant m} \partial_\xi^\alpha T_\xi^- h_\alpha - T_\xi^- h^I.$$

We then localize with a test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}_+^* \times \mathbb{R}_x^N)$:

$$\begin{aligned}\phi Q_\xi^\lambda u = {} & \lambda \phi f - T_\xi^-\big(\phi h^I\big) + \big(T_\xi^- \phi\big) h^I \\ & - \sum_{|\alpha| \leqslant m} \big[T_\xi^-, \partial_\xi^\alpha\big](\phi h_\alpha) + \sum_{|\alpha| \leqslant m} \big[T_\xi^-, \partial_\xi^\alpha\big](\phi) h_\alpha \\ & - \sum_{|\alpha| \leqslant m} \partial_\xi^\alpha T_\xi^-(\phi h_\alpha) + \sum_{|\alpha| \leqslant m} \partial_\xi^\alpha \big(T_\xi^- \phi\big) h_\alpha.\end{aligned}$$

Besides we get:

$$\begin{aligned}Q_\xi^\lambda(\phi u) = {} & \phi Q_\xi^\lambda u + \big(T^+ T^- \phi - \lambda \Box_{t,x} \phi\big) u \\ & + 2\lambda \partial_t (u \partial_t \phi) - 2\lambda \nabla_x \cdot (u \nabla_x \phi) - T^-\big(u T^+ \phi\big) - T^+\big(u T^- \phi\big).\end{aligned}$$

Gathering the two last equations we find that $Q_\xi^\lambda(\phi u)$ can be written as a sum:

$$Q_\xi^\lambda(\phi u) = \lambda \phi f + \sum_i \partial_\xi^{\alpha_i} \partial_{t,x}^{\beta_i} w_i,$$

with new multiindex $|\alpha_i| \leqslant m$ and $|\beta_i| \leqslant 1$, and functions $w_i \in L^1(\mathbb{R}_\xi^M, W^{\gamma,p}(\mathbb{R}_+^* \times \mathbb{R}_x^N))$ (as shown by Lemma 2.2). We apply the Fourier transform in the variables (t,x) to the previous equality and denote by (ω,k) the corresponding Fourier variables.

$$\big(q_\xi^\lambda \widehat{\phi u}\big)(\omega,k,\xi) = \lambda \widehat{\phi f}(\omega,k,\xi) + \sum_i \partial_\xi^{\alpha_i} \big(\widehat{\partial_{t,x}^{\beta_i} w_i}\big)(\omega,k,\xi).$$

Pick a test function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}_\xi^M)$ and fix λ such that condition (2.3) of Lemma 2.1 holds. Choose also a cut-off function $\theta \in \mathcal{C}^\infty(\mathbb{R}_\omega \times \mathbb{R}_k^N)$ that vanishes near the origin and equals 1 outside a sufficiently large ball. Averaging in ξ in the sense of distributions, we find:

$$\begin{aligned}&\theta \int \widehat{\phi u}\, \chi(\xi)\, d\xi \\ &\quad = \theta \int \lambda \widehat{\phi f} \left(\frac{\chi}{q_\xi^\lambda}\right)(\xi)\, d\xi + \sum_i (-1)^{|\alpha_i|} \theta \int \widehat{\partial_{t,x}^{\beta_i} w_i}\, \partial_\xi^{\alpha_i} \left(\frac{\chi}{q_\xi^\lambda}\right)(\xi)\, d\xi. \qquad (2.11)\end{aligned}$$

Distributing the derivative $\partial_\xi^{\alpha_i}$, we write

$$\partial_\xi^{\alpha_i}\left(\frac{\chi}{q_\xi^\lambda}\right)=\sum_{\nu_i+\delta_i=\alpha_i} C(\alpha_i,\delta_i)\partial_\xi^{\delta_i}\left(\frac{1}{q_\xi^\lambda}\right)\partial_\xi^{\nu_i}\chi,$$

with $C(\alpha_i,\delta_i)\in\mathbb{N}$. We define ρ_i such that

$$\widehat{\rho_i}=(-1)^{|\alpha_i|}\theta\int\partial_\xi^{\delta_i}\left(\frac{1}{q_\xi^\lambda}\right)\widehat{\partial_{t,x}^{\beta_i}w_i}\,\partial_\xi^{\nu_i}\chi(\xi)\,d\xi,$$

we have to show that each ρ_i belongs to $W^{1+\gamma,p}(\mathbb{R}^{1+N})$. Look at one of these terms and abandon the subscript i:

$$\widehat{\rho}=(-1)^{|\alpha|}\theta\int\partial_\xi^{\delta}\left(\frac{1}{q_\xi^\lambda}\right)\widehat{\partial_{t,x}^{\beta}w}\,\partial_\xi^{\nu}\chi(\xi)\,d\xi=\int m\widehat{w}\partial_\xi^\nu\chi(\xi)\,d\xi,$$

where $m=\theta\partial_\xi^\delta(1/q_\xi^\lambda)n$ with n a homogeneous polynomial of order 0 or 1 in the variables (ω,k). It remains to show there is a mapping:

$$\begin{aligned}T: L^1\big(\mathbb{R}_\xi^M, W^{\gamma,p}\big(\mathbb{R}^{1+N}\big)\big) &\longrightarrow W^{1+\gamma,p}\big(\mathbb{R}^{1+N}\big),\\ w&\longmapsto \rho.\end{aligned}$$

We write

$$\big(1+\omega^2+|k|^2\big)^{(1+\gamma)/2}\widehat{\rho}(\omega,k)=\int m_1(\omega,k,\xi)\widehat{w_\gamma}(\omega,k,\xi)\partial_\xi^\nu\chi(\xi)\,d\xi,$$

with $m_1(\omega,k,\xi)=(1+\omega^2+|k|^2)^{1/2}m(\omega,k,\xi)$ and $w_\gamma=(I-\Delta_{t,x})^{\gamma/2}w$. We have $w_\gamma\in L^1_{\mathrm{loc}}(\mathbb{R}_\xi^M, L^p(\mathbb{R}^{1+N}))$, so let us see that $m_1(\cdot,\cdot,\xi)$ satisfies uniformly in ξ the assumptions of Theorem 2.1. Establish these estimates for m_1:

$$\partial_{\omega,k}^\alpha m_1(\omega,k,\xi)=\sum_{\beta+\delta=\alpha} C(\alpha,\beta)\partial_{\omega,k}^\beta\big(1+\omega^2+|k|^2\big)^{1/2}\partial_{\omega,k}^\delta m(\omega,k,\xi),$$

with $C(\alpha,\beta)\in\mathbb{N}$. Remember that $m=\theta\partial_\xi^\nu(1/q_\xi^\lambda)n$ for some ν and n is a polynomial of order 0 or 1. Since θ vanishes near the origin, the estimates provided in Lemma 2.1 ensures that

$$\sup_{\xi\in\operatorname{supp}\chi}\ \sup_{(\omega,k)\in\operatorname{supp}\theta}\big(\omega^2+|k|^2\big)^{(1+|\delta|)/2}\left|\partial_{\omega,k}^\delta\left(\theta\partial_\xi^\nu\left(\frac{1}{q_\xi^\lambda}\right)n\right)(\omega,k,\xi)\right|<+\infty,$$

and it is clear that

$$\sup_{\xi\in\operatorname{supp}\chi}\ \sup_{(\omega,k)\in\operatorname{supp}\theta}\big(\omega^2+|k|^2\big)^{(|\beta|-1)/2}\big|\partial_{\omega,k}^\beta\big(1+\omega^2+|k|^2\big)^{1/2}\big|<+\infty.$$

Thus we get for any α,

$$\sup_{\xi\in\operatorname{supp}\chi}\ \sup_{(\omega,k)\in\operatorname{supp}\theta}\big(\omega^2+|k|^2\big)^{|\alpha|/2}\big|\partial_{\omega,k}^\alpha m_1(\omega,k,\xi)\big|<+\infty.$$

For we only need to consider the cases $|\alpha|\leqslant l$, where l is an integer such that $l>N/2$, we deduce that Theorem 2.1 applies uniformly in ξ and eventually that $\rho\in W^{1+\gamma,p}(\mathbb{R}^{1+N})$.

We return to (2.11). With the same argument for the remaining term $\theta \int \lambda \widehat{\phi f}(\chi/q_\xi^\lambda)(\xi)\,d\xi$, we find that $\theta \int \widehat{\phi u}\chi(\xi)\,d\xi$ is the Fourier transform of a function in $W^{1+\gamma,p}(\mathbb{R}^{1+N})$. Now observe that the function $\int \widehat{\phi u}\chi(\xi)\,d\xi$ is smooth, as the Fourier transform of a compactly supported distribution. But since $1-\theta$ is nothing but a test function in $\mathcal{C}_c^\infty(\mathbb{R}^{1+N})$, it follows that

$$(1-\theta)\int \widehat{\phi u}\chi(\xi)\,d\xi \in \mathcal{C}_c^\infty\big(\mathbb{R}^{1+N}\big).$$

Thus we can conclude that $\int u\chi\,d\xi$ belongs to $W_{\mathrm{loc}}^{1+\gamma,p}(\mathbb{R}_+^* \times \mathbb{R}_x^N)$.

3. Proof of Theorem 1.2

3.1. A division lemma

Let $Y \in \mathcal{D}'(\mathbb{R}^4)$ be the forward elementary solution of the d'Alembertian:

$$Y(t,x) = \frac{\mathbf{1}_{t>0}}{4\pi t}\delta_{|x|=t}.$$

To each $v \in \mathbb{R}^3$ is associated the streaming operator $T = \partial_t + v\cdot\nabla_x$. Let $\mathcal{M}_m$ be the space of C^∞ homogeneous functions of degree m on $\mathbb{R}^4 \setminus 0$. Below, we use the notation

$$x_0 := t, \quad \text{and} \quad \partial_j := \partial_{x_j}, \quad j = 0,\dots,3. \tag{3.1}$$

The result given now is a part of the division lemma in [2]:

Lemma 3.1. *For each $v \in \mathbb{R}^3$ such that $|v| < 1$, there exists functions $a_i^k \equiv a_i^k(t,x)$ where $i = 0,\dots,3$ and $k = 0,1$, such that $a_i^k \in \mathcal{M}_{-k}$ and*

$$\partial_i Y = T\big(a_i^0 Y\big) + a_i^1 Y, \quad i = 0,\dots,3. \tag{3.2}$$

In formula (3.2), the expressions $a_i^0 Y$ and $a_i^1 Y$ designate, for each $i = 0,\dots,3$, the unique homogeneous distributions on $\mathbb{R}^4$ whose restrictions to $\mathbb{R}^4 \setminus 0$ are the expressions above.

Proof. Full details can be found in [2]. Here we just recall how to choose the functions a_i^k. Set

$$\alpha_0(t,x) = \frac{t}{t - x\cdot v}, \qquad \alpha_j(t,x) = \frac{x_j}{x\cdot v - t}, \quad j = 1,2,3.$$

Then (3.2) holds with:

$$a_j^0(t,x) = \alpha_j(t,x)\chi\left(\frac{|x|}{t}\right), \quad \text{and} \quad a_j^1 = -Ta_j^0, \quad j = 0,\dots,3, \tag{3.3}$$

where $\chi \in C_c^\infty(\mathbb{R}_+)$ satisfies

$$0 \leqslant \chi \leqslant 1, \quad \chi|_{[0,1/2+1/(2|v|)]} \equiv 1, \quad \operatorname{supp}\chi \subset \left[0, \frac{1}{|v|}\right[.$$

Notice that since $|v| < 1$, we have that $a_i^k \in \mathcal{M}_{-k}$ as expected. □

3.2. Proof

We shall assume here that $p=+\infty$, since the arguments in the case $p=1$ are the same. The solution of the Cauchy problem for the wave equation in (1.1) with initial data (u_I, u_I') reads[2]

$$u(t,x,\xi)=\partial_t\big(Y(t,\cdot)\star_x u_I\big)(x,\xi)+\big(Y(t,\cdot)\star_x u_I'\big)(x,\xi)+(Y\star f)(t,x,\xi).$$

In view of the assumptions made on u_I and u_I', it is obvious that the two first terms satisfy the conclusion of the theorem. Now we consider the case where the initial data vanish. It is no doubt that $\int u\chi\,d\xi\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+^*\times\mathbb{R}_x^3)$, so now look at the derivatives. Using the lemma established above we can write

$$\partial_i u=T\big(a_i^0 Y\big)\star(f\mathbf{1}_{t>0})+\big(a_i^1 Y\big)\star(f\mathbf{1}_{t>0}),$$

where $a_i^k\equiv a_i^0(t,x,\xi)$ are given by (3.3) for $v\equiv v(\xi)$. Observe that these coefficients belong to $\mathcal{C}^\infty(\mathbb{R}^4\backslash\{0\}\times\mathbb{R}^M)$ and that $\partial_\xi^\beta a_i(\cdot,\cdot,\xi)\in\mathcal{M}_{-k}$ for any $\xi\in\mathbb{R}^M$ and any multiindex $\beta\in\mathbb{N}^M$. Using the second equation in (1.1), it comes

$$T(f\mathbf{1}_{t>0})=(Tf)\mathbf{1}_{t>0}+f\delta_{t=0}=\sum_{|\alpha|\leqslant m}\partial_\xi^\alpha(a_\alpha g\mathbf{1}_{t>0})+f\delta_{t=0}.$$

Let $\chi\in\mathcal{C}_c^\infty(\mathbb{R}_\xi^M)$ be a test function. Then

$$\begin{aligned}\partial_i\int u(t,x,\xi)\chi(\xi)\,d\xi&=\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}\int\big(\alpha_i^0 Y\big)\star(a_\alpha g\mathbf{1}_{t>0})\partial_\xi^\alpha\chi(\xi)\,d\xi\\&\quad+\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}\int\big(\partial_\xi^\alpha\alpha_i^0 Y\big)\star(a_\alpha g\mathbf{1}_{t>0})\chi(\xi)\,d\xi\\&\quad+\int\big(\alpha_i^0 Y\big)\star_x f_I\chi(\xi)\,d\xi+\int\big(\alpha_i^1 Y\big)\star(f\mathbf{1}_{t>0})\chi(\xi)\,d\xi.\end{aligned}$$

Now we bound each terms in the right-hand side. For any function $h\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+^*\times\mathbb{R}^3\times\mathbb{R}^M)$, we have

$$\big(\partial_\xi^\beta\alpha_i^k Y\big)\star h(t,x,\xi)=\int_0^t\int_{|y|=s}\partial_\xi^\beta\alpha_i^k(s,y,\xi)h(t-s,x-y,\xi)\,d\sigma_y\,\frac{ds}{4\pi s},$$

therefore

$$\big|\big(\partial_\xi^\beta\alpha_i^k Y\big)\star h(t,x,\xi)\big|\leqslant C\big(1+t^2\big)\int_{|y|=1}\big|\partial_\xi^\beta\alpha_i^k(1,y,\xi)\big|\,d\sigma_y\big\|h(\cdot,\cdot,\xi)\big\|_{L^\infty(C(t,x))},$$

with $C(t,x)=\{(s,y)\colon |x-y|\leqslant t-s\}$. We do the same with the term involving the initial data, and we get

$$\bigg\|\partial_i\int u(t,x,\xi)\chi(\xi)\,d\xi\bigg\|_{L^\infty(K)}\leqslant C(K),$$

[2] $\star$ denotes convolution in (t,x).

for any compact $K \subset \mathbb{R}_+^* \times \mathbb{R}^3$.

4. Counterexamples

4.1.

Here we construct in the case $m = 0$ an example showing that the nonresonant condition is crucial. Below we assume $N \geqslant 2$, but a similar counterexample can be constructed when $N = 1$.

Proposition 4.1. *Let $\varepsilon > 0$ and $v(\xi) \equiv (1, 0, \dots, 0)$. Then there exists $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}_+^* \times \mathbb{R}^N)$ such that $u \notin W^{1+\varepsilon,p}_{\mathrm{loc}}(\mathbb{R}_+^* \times \mathbb{R}^N)$ but satisfying, with vanishing initial data,*

$$\Box_{t,x} u = f,$$
$$\partial_t f + v \cdot \nabla_x f = g,$$

for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}^N$ and with $f, g \in L^p_{\mathrm{loc}}(\mathbb{R}_+^* \times \mathbb{R}^N)$.

Proof. Pick $\psi_1 \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\psi_2 \in \mathcal{C}_c^\infty(\mathbb{R}^{N-1})$ two test functions and consider the application L defined for any $\phi \in L^p(\mathbb{R})$ by

$$(L\phi)(t, x_1, x_2, \dots, x_N) = \phi\left(\frac{t - x_1}{\sqrt{2}}\right)\psi_1\left(\frac{t + x_1}{\sqrt{2}}\right)\psi_2(x_2, \dots, x_N).$$

Let us see that L is a bounded operator that maps $W^{s,p}(\mathbb{R})$ into $W^{s,p}(\mathbb{R}^{1+N})$. This is true when $s = 0$. Indeed the equality

$$\|L\phi\|_{L^p(\mathbb{R}^{1+N})} = \|\phi\|_{L^p(\mathbb{R})}\|\psi_1\|_{L^p(\mathbb{R})}\|\psi_2\|_{L^p(\mathbb{R}^{1+N})}$$

holds for any p. Similarly this is also true for integral values of s. L is injective so we can define $L^{-1} : \operatorname{Im} L \to W^{s,p}(\mathbb{R})$. The open mapping theorem then shows that L^{-1} is also bounded and we have for any integer s:

$$C_1\|\phi\|_{W^{s,p}(\mathbb{R})} \leqslant \|L\phi\|_{W^{s,p}(\mathbb{R}^{1+N})} \leqslant C_2\|\phi\|_{W^{s,p}(\mathbb{R})}.$$

Interpolating between the integral values we find that the previous inequalities still hold for any real $s > 0$. As a consequence the operator $L : W^{s,p}(\mathbb{R}) \to W^{s,p}(\mathbb{R}^{1+N})$ is well-defined for any nonnegative s and satisfy the additional property

$$\phi \in W^{s,p}(\mathbb{R}) \quad \text{if and only if} \quad L\phi \in W^{s,p}\big(\mathbb{R}^{1+N}\big). \tag{4.1}$$

Now pick a compactly supported function $\phi \in W^{1,p}(\mathbb{R}) \setminus W^{1+\varepsilon,p}(\mathbb{R})$ such that $\operatorname{supp} \phi \subset \mathbb{R}_-^*$. Choose also a test function $\psi_1 \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfying $\operatorname{supp} \psi_1 \subset \mathbb{R}_+$. Define

$$u(t, x_1, \dots, x_N) = \phi\left(\frac{t - x_1}{\sqrt{2}}\right)\psi_1\left(\frac{t + x_1}{\sqrt{2}}\right).$$

Then $\operatorname{supp} u \subset [\delta, +\infty) \times \mathbb{R}^3$ for some $\delta > 0$. Clearly,

$$\Box_{t,x} u = \big(\partial_t^2 - \partial_{x_1}^2\big) u(t, x_1, \dots, x_N) = \phi'\left(\frac{t - x_1}{\sqrt{2}}\right)\psi_1'\left(\frac{t + x_1}{\sqrt{2}}\right).$$

If we note

$$f(t,x_1,\dots,x_N)=\phi'\left(\frac{t-x_1}{\sqrt{2}}\right)\psi_1'\left(\frac{t+x_1}{\sqrt{2}}\right),$$

we have also

$$(\partial_t+v\cdot\nabla_x)f=(\partial_t+\partial_{x_1})f(t,x_1,\dots,x_N)=\phi'\left(\frac{t-x_1}{\sqrt{2}}\right)\psi_1''\left(\frac{t+x_1}{\sqrt{2}}\right).$$

We set

$$g(t,x_1,\dots,x_N)=\phi'\left(\frac{t-x_1}{\sqrt{2}}\right)\psi_1''\left(\frac{t+x_1}{\sqrt{2}}\right).$$

It remains to see that u, f and g satisfy the desired properties. Let $\psi\in\mathcal{C}_c^\infty(\mathbb{R}_+^*\times\mathbb{R}^N)$ be an arbitrary test function. Then one can find $\psi_2\in\mathcal{C}_c^\infty(\mathbb{R}^{N-1})$ such that

$$(\psi u)(t,x_1,\dots,x_N)=(\psi u)(t,x_1,\dots,x_N)\psi_2(x_2,\dots,x_N).$$

So we can write $\psi u=\psi(L\phi)$, with L defined as above. We infer from (4.1) that $\psi u\in W^{1,p}(\mathbb{R}^{1+N})\setminus W^{1+\varepsilon,p}(\mathbb{R}^{1+N})$. For the same reason we find that ψf and ψg belong to $L^p(\mathbb{R}^{1+N})$ since $\phi'\in L^p(\mathbb{R})$. $\square$

4.2.

The exponent $\gamma=1-(N-1)|\frac{1}{2}-\frac{1}{p}|$ is the best one can expect under the assumptions of Theorem 1.1, because the regularity result for the wave equation is optimal.

Proposition 4.2. *Let $\varepsilon>0$ small and assume $1<p<+\infty$ is such that $(N-1)|\frac{1}{2}-\frac{1}{p}|\geqslant 0$. There exists $u\in W^{1+\gamma,p}_{\mathrm{loc}}(\mathbb{R}_+^*\times\mathbb{R}^N)$ satisfying $u\notin W^{1+\gamma+\varepsilon,p}_{\mathrm{loc}}(\mathbb{R}_+^*\times\mathbb{R}^N)$ and such that*

$$\Box_{t,x}u=0,$$

with initial data

$$u_{|t=0}=0,\qquad \partial_t u_{|t=0}=u_I\in W^{1,p}_{\mathrm{loc}}\big(\mathbb{R}^N\big).$$

Proof. By an argument of duality, we can assume that $p\leqslant 2$. Consider a function $\phi_r\in\mathcal{C}^\infty(\mathbb{R}^N\setminus\{0\})$ such that

$$\phi_r(x)=\frac{1}{|x|^r}\quad\text{near zero,}$$

and with a good behaviour at infinity. We choose $r=1/p+(N+1)/2-\gamma-\varepsilon$ so that ϕ_r belongs to $L^p(\mathbb{R}^N)$ and define u as the solution to the Cauchy problem for the wave equation $\Box_{t,x}u=0$ with initial conditions

$$u_{|t=0}=0,\qquad \partial_t u_{|t=0}=\phi_r.$$

Then it is known [3] that

$$u(1,\cdot)\notin W^{\gamma+\varepsilon,p}\big(\mathbb{R}^N\big).$$

So if we replace ϕ_r with $(I - \Delta_x)^{-1/2}\phi_r$ then obviously

$$u(1,\cdot) \in W^{1+\gamma,p}\big(\mathbb{R}^N\big) \setminus W^{1+\gamma+\varepsilon,p}\big(\mathbb{R}^N\big).$$

With the argument used in Section 2.3, we get the desired result. $\square$

References

[1] F. Bouchut, F. Golse, C. Pallard, Non-resonant smoothing for coupled wave + transport equations and the Vlasov–Maxwell system, Rev. Mat. Iberoamericana, 2003, in press.
[2] F. Bouchut, F. Golse, C. Pallard, On classical solutions to the 3D relativistic Vlasov–Maxwell system: Glassey–Strauss' theorem revisited, Arch. Rational Mech. Anal., in press.
[3] J.C. Peral, L^p estimates for the wave equation, J. Funct. Anal. 36 (1) (1980) 114–145.
[4] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, New Jersey, 1970.